Poisson-Voronoi tessellations and fixed price for higher rank lattices

McGill DDC Seminar

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9 May 2023

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Background

History of the IPVT

Budzinski, Curien, Petri (2022): description of the pointless Voronoi tessellation on $\mathbb{H}^2$

D'Achille, Curien, Enriquez, Lyons, Unel (2023): construction of the ideal Poisson-Voronoi tessellation (IPVT) on $\mathbb{H}^d$

Fraczyk, Mellick, Wilkens (soon): construction of the IPVT on a higher rank real semisimple Lie group G

Π_η = Poisson point process on X with intensity η

Idea for $\mathbb{H}^2$

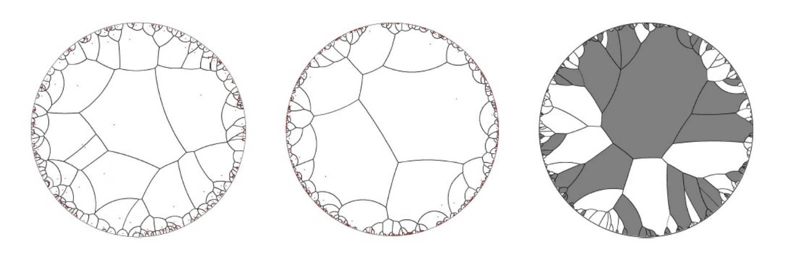


Figure from: Thomas Budzinski, Nicolas Curien, and Bram Petri, *On Cheeger constants of hyperbolic surfaces*, arXiv e-prints (2022), arXiv:2207.00469.

IPVT construction

$G =$ could be $\mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$

lcsc csms

Horocones

We call the object on which the IPVT lives a horocone.

The horocone for $G = SL(2, \mathbb{R})$ and $X = \mathbb{H}^2$ is G modded out by the subgroup of upper triangular matrices with ones on the diagonal, equivalently $\partial X \times \mathbb{R}$, equipped with Lebesgue measure.

U

$X = G/K$

cost of Poi on each is the same

Theorem (FMW)

Any nonamenable locally compact second countable (lcsc) group has a horocone. $G \curvearrowright G$

For a semisimple real Lie group G the horocone is G/U, equivalently $\partial X \times \mathbb{R}$, equipped with a G-invariant measure unique up to scaling.

$G = SL(n, \mathbb{R})$, P = minimal parabolic subgp of G, $U \leq P$ max unimod. in P

Horocone construction

Fix a basepoint $o \in X$. Let d be a G-invariant metric on X and m a G-invariant measure on X. Define the space of "distance-like" functions on X as

we want to get Busemann functions / points on ∂X

$$D := \mathrm{cl}\{x \mapsto d(x,y) + t | y \in X, t \in \mathbb{R}\} \subseteq \mathcal{C}(X).$$

We have $G \curvearrowright D$ with $gf(x) := f(g^{-1}x)$.

G-eq. embedding

For $t \in \mathbb{R}$, define $\iota_t : X \to D$ by $\iota_t(x)(y) = d(x,y) - t$, where $y \in X$.

normalization

Let $\eta_t := m(B(o,t))^{-1}$ ($t \to \infty \Leftrightarrow \eta_t \to 0$).

Set $\underline{\mu_t} := \eta_t(\iota_t)_*(m)$. normalized push-forward of m under ι_t

Goal: G-inv. measure on D

Horocone construction, continued

The sequence of measures $\{\mu_t\}_{t\in\mathbb{R}}$ has a non-zero subsequential weak-$*$ limit μ as $t \to \infty$ whenever (X, d) has exponential growth.

In particular, such a μ exists for any nonamenable lcsc group.

Then μ is our desired G-invariant measure on D, and (D, μ) is the horocone for G.

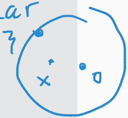
$\gamma' : [0, \infty) \to X$ another one

$\gamma : [0, \infty) \to X$ parameterization of geodesic ray from O to ξ

Geometric intuition

Consider $X = \mathbb{H}^2$ and a boundary point $\xi \in \partial X$.

∂X can be identified with G/P where $G = SL(2, \mathbb{R})$ and P is the minimal parabolic subgroup of G. upper triangular matrices



Define $\beta_\gamma(x) := \lim_{t\to\infty} d(x, \gamma(t)) - d(o, \gamma(t)) \in D$.

The boundary $\partial X = G/P$ is the corresponding equivalence class of Busemann functions. $\beta_{\gamma'}(x)$ and $\beta_\gamma(x)$ differ by a constant

no G-inv. measure

Without equivalence, we end up with $\partial X \times \mathbb{R} = G/U$, where U is the maximal unimodular subgroup of P.

Horocones and the IPVT

The G-invariant measure μ on D determines the Poisson point process on D:

The limit $\lim_{t\to\infty}\Pi_{\eta_t}$ where each Π_{η_t} is a Poisson point process on X with intensity η_t converges to a Poisson point process on the horocone G/U with positive intensity.

For $x \in X$, if

$\beta_{gU} \in G/U$

$$\beta_{gU}(x) = \min\{\beta_{hU}(x) | hU \text{ belongs to the Poisson on } G/U\}$$

then x lives in the IPVT cell of gU.

Cost review

How to prove G and its lattices have fixed price one

Use the following theorems from Abert, Mellick (2021):

The Poisson point process action on G has maximal cost out of all essentially free, measure-preserving actions on G.

Let Π be a Poisson point process on G and D a complete and separable metric space with a G-action. Suppose $\Phi_t(\Pi)$ is a sequence of measurable and equivariant D-valued factors of Π such that $\Phi_t(\Pi)$ weakly converges to a random process Υ on D. Then Π and $\Pi \times \Upsilon$ have the same cost.

If G has fixed price one, then so does any lattice in G.

Unbounded walls

Theorem (FMW)

For a higher rank real semisimple Lie group G, each pair of cells in its IPVT almost-surely share an unbounded wall.

Sketch of the proof

Let Π be the Poisson point process on G/U associated to the IPVT on X. Fix any two points belonging to Π; call them g_1U, g_2U. Define $W(r)$ to be set of points $x \in X$ such that:

$r > 0$

$\beta_{g_1U}(x) = \beta_{g_2U}(x)$

$x \in W(r)$ belongs to bdy shared by cells of g_1U, g_2U

and

$\beta_{gU}(x) > \beta_{g_1U}(x) + r$ for every $gU \in \Pi \setminus \{g_1U, g_2U\}$.

$1/2\, r$ nbd of x only sees cells of g_1U, g_2U

Sketch of the proof, continued

Define $W(r)$ to be set of points $x \in X$ such that:

$\beta_{g_1 U}(x) = \beta_{g_2 U}(x)$ *

and

$\beta_{gU}(x) > \beta_{g_1 U}(x) + r$ for every $gU \in \Pi \setminus \{g_1 U, g_2 U\}$.

Claim: $W(r)$ is almost-surely unbounded.

*

We start with $x \in X$ such that $\beta_{g_1 U}(x) = \beta_{g_2 U}(x)$. Then we produce an unbounded set contained in $W(r)$ from an action on x.

Sketch of the proof, continued

stabs g_1U $\to g_2U$

The stabilizer subgroup $S := g_1Ug_1^{-1} \cap g_2Ug_2^{-1}$ fixes g_1U, g_2U but mixes up almost every other point of Π.

$S = \mathbb{R}^{\operatorname{rank} G - 1} \times$ compact subgp

S is non-compact only when G is higher rank.

Howe-Moore implies $\lim_{i\to\infty} \mu(B \cap s_iB) = 0$ for Borel $B \subseteq G/U$ and any escaping sequence $\{s_i\}_{i\in\mathbb{N}} \subseteq S$.

Set $B := \{gU \in G/U : \beta_{gU}(x) < \beta_{g_1U}(x) + r\}$. $\mu(B) < \infty$

The set of points in G/U that are "closer" to x than $\beta_{g_1U}(x)+r$

As a consequence of the horocone construction, $\mu(B) < \infty$.

Sketch of the proof, continued

unbounded

By Howe-Moore, there exists a subsequence $\{s_{i_j}\} \subseteq \{s_i\}$ such that for large enough $j \ll k$, $\mu(s_{i_j}B \cap s_{i_k}B)$ is arbitrarily small.

these sets are not disjoint

$\{\Pi(B) = 0\}$ has positive probability

Let E_j be the event $\{\Pi(s_{i_j}B) = 0\}$.

We can apply a version of Borel-Cantelli to conclude the E_j occur infinitely often almost-surely.

(r) $W(r)$

For each E_j which occurs, we have $s_{i_j}^{-1}x \in W^r_\Pi$. So W^r_Π is unbounded almost-surely.